1.1. Since the plastic strain is one quarter of the elastic strain, the total strain at the yield point is

$$\varepsilon = \varepsilon^e + \varepsilon^p = \frac{5}{4}\,\varepsilon^e = \frac{5Y}{4E}$$

and the stress/strain equation gives

$$Y = \frac{E}{180}\left(\frac{5Y}{4E}\right)^{0.25}$$

or

$$\left(\frac{Y}{E}\right)^{0.75} = \frac{1}{180}\left(\frac{5}{4}\right)^{0.25}$$

or

$$\frac{Y}{E} = \left(\frac{1}{180}\right)^{4/3} \left(\frac{5}{4}\right)^{1/3} = \frac{1}{180}\left(\frac{1}{144}\right)^{1/3} \cong \frac{1}{943}\ .$$

The true strain at instability is $\varepsilon = 0.25$, while the true and nominal stresses at instability are given by

$$\frac{\sigma}{Y} = \frac{E}{Y}\,\frac{(0.25)^{0.25}}{180} = (144)^{1/3}\,(0.25)^{0.25} \cong 3.71$$

$$\frac{s}{Y} = \frac{\sigma}{Y}\left(\frac{A}{A_0}\right) = \frac{\sigma}{Y}\,e^{-\varepsilon} = 3.71\,e^{-0.25} \cong 2.89\ .$$

1.2. The engineering strain at the onset of instability is given by the instability condition

$$\frac{d\sigma}{de} = \frac{d}{de}\,(C\,e^n) = \frac{n\,\sigma}{e} = \frac{\sigma}{1+e}$$

or $\quad (1+e)n = e\ ,\quad$ or $\quad e = n/(1-n)$

and $\quad \varepsilon = \ell n(1+e) = \ell n\left(\frac{1}{1-n}\right)$, which exceeds $\ell n\left(\frac{1}{1-e}\right)$ when $e < n$.

Since the magnitude of the total true strain at instability must be equal to $\ell n\{1/(1-n)\}$,

$$\ell n\,\frac{1}{1-e} + \ell n\,\frac{\ell_2}{\ell_1} = \ell n\left(\frac{1}{1-n}\right)$$

where ℓ_1 is the length of the bar at the end of compression, and ℓ_2 the length at instability. Then

$$\ell_2 = \ell_1\left(\frac{1-e}{1-n}\right) = \ell_0(1-e)\left(\frac{1-e}{1-n}\right) = \ell_0\,\frac{(1-e)^2}{1-n}$$

where ℓ_0 is the initial length of the bar.

1.3. According to the Voce equation, the instability condition is

$$\frac{d\sigma}{d\varepsilon} = \frac{d}{d\varepsilon}\{C(1-m\,e^{-n\varepsilon})\} = C\,m\,n\,e^{-n\varepsilon} = n(C-\sigma)$$

and the onset of instability corresponds to

$$n(C-\sigma) = \sigma, \quad \text{or} \quad \sigma = Cn/(1+n) \ .$$

Hence, $$e^{n\varepsilon} = \frac{C\ m}{C-\sigma} = m\left(1 - \frac{n}{1+n}\right) = m(1+n)$$

and $$\varepsilon = \frac{1}{n}\,\ell n\,[\,m(1+n)\,] \ , \quad m(1+n) \gtrsim 0 \ .$$

When $m(1+n) < 1$, the instability strain is zero for a rigid/plastic bar. Introducing ε^*,

$$\sigma = C(1-m\ e^{-n\varepsilon}) = C\left\{1-m\left(\frac{\ell_o}{\ell}\right)^n\right\}$$

$$= C\left\{(1-m)+m\left[1-\left(\frac{\ell_o}{\ell}\right)^n\right]\right\} = C(1-m+mn\varepsilon^*) \ .$$

The stress-strain curve is thus linearized. Note that when $n=0$, $\varepsilon^* = n(\ell/\ell_o) = \varepsilon$.

1.4. The total compressive load at any stage is

$$P = \sigma\ A = \sigma\ A_o\left(\frac{h_o}{h}\right) = \sigma\ A_o/(1-e) \ .$$

Since $d\varepsilon = de/(1-e)$ in compression, the differentiation of the above equation gives

$$\frac{dP}{de} = \frac{A_o}{(1-e)^2}\,\frac{d\sigma}{d\varepsilon} + \frac{\sigma\ A_o}{(1-e)^2} = \frac{A_o}{(1-e)^2}\left(\frac{d\sigma}{d\varepsilon} + \sigma\right)$$

$$\frac{d^2P}{de} = \frac{A_o}{(1-e)^3}\left(\frac{d^2\sigma}{d\varepsilon^2} + \frac{d\sigma}{d\varepsilon}\right) + \frac{2A_o}{(1-e)^3}\left(\frac{d\sigma}{d\varepsilon} + \sigma\right) .$$

Hence, at the point of inflection ($d^2P/de^2 = 0$),

$$\frac{d^2\sigma}{d\varepsilon^2} + 3\,\frac{d\sigma}{d\varepsilon} + 2\sigma = 0 \ .$$

The empirical stress-strain equation $\sigma = C\varepsilon^n$ gives

$$\frac{d\sigma}{d\varepsilon} = \frac{n\ \sigma}{\varepsilon} \ , \quad \frac{d^2\sigma}{d\varepsilon^2} = \frac{n}{\varepsilon}\,\frac{d\sigma}{d\varepsilon} - \frac{n\ \sigma}{\varepsilon^2} = n(n-1)\,\frac{\sigma}{\varepsilon^2} \ .$$

Hence the point of inflection corresponds

$$\left\{\frac{n(n-1)}{\varepsilon^2} + \frac{3n}{\varepsilon} + 2\right\}\sigma = 0$$

or $$2\varepsilon^2 + 3\,n\,\varepsilon - n(1-n) = 0$$

or $$\varepsilon = \frac{1}{4}\,[\,-3n + \sqrt{n(8+n)}\,]$$

Thus, $\varepsilon > n$ corresponds to $\sqrt{n(8+n)} > 7n$, or $8n + n^2 > 49n^2$, or $n < 1/6$.

1.5. The nominal compressive stress at any stage is

$$s = \frac{P}{A_o} = \frac{P}{A}\left(\frac{h_o}{h}\right) = \sigma \exp(\varepsilon) .$$

By successive differentiation, we get

$$\frac{ds}{d\varepsilon} = \left(\frac{d\sigma}{d\varepsilon} + \sigma\right) \exp(\varepsilon)$$

$$\frac{d^2 s}{d\varepsilon^2} = \left(\frac{d^2\sigma}{d\varepsilon^2} + 2\frac{d\sigma}{d\varepsilon} + \sigma\right) \exp(\varepsilon) .$$

The point of inflection corresponds to $d^2s/d\varepsilon^2 = 0$, or

$$\frac{d^2\sigma}{d\varepsilon^2} + 2\frac{d\sigma}{d\varepsilon} + \boldsymbol{\sigma} = \left(\frac{d}{d\varepsilon} + 1\right)^2 \sigma = 0$$

or $$\left\{\frac{n(n-1)}{\varepsilon^2} + \frac{2n}{\varepsilon} + 1\right\}\sigma = 0$$

or $$\varepsilon^2 + 2n\varepsilon - n(1-n) = 0, \quad \text{or} \quad \varepsilon = \sqrt{n} - n.$$

The inflection strain will exceed the uniaxial instability strain if $\sqrt{n} - n > n$, or $n < 0.25$.

1.6. The differentiation of the Ramberg-Osgood equation furnishes

$$\frac{d\varepsilon}{d\sigma} = \frac{1}{E} + \frac{3}{7nE}\left(\frac{\sigma}{\sigma_o}\right)^{(1-n)/n} = \frac{1}{\sigma}$$

at the onset of instability in simple tension.

Since $E >> \sigma$, we have

$$\frac{3\sigma_o}{7nE}\left(\frac{\sigma}{\sigma_o}\right)^{1/n} \cong 1, \quad \text{or} \quad \frac{\sigma}{\sigma_o} \cong \left(\frac{7nE}{3\sigma_o}\right)^n$$

and $$\varepsilon \cong \frac{\sigma_o}{E}\left(\frac{7nE}{3\sigma_o}\right)^n + \frac{3\sigma_o}{7E}\left(\frac{7nE}{3\sigma_o}\right)$$

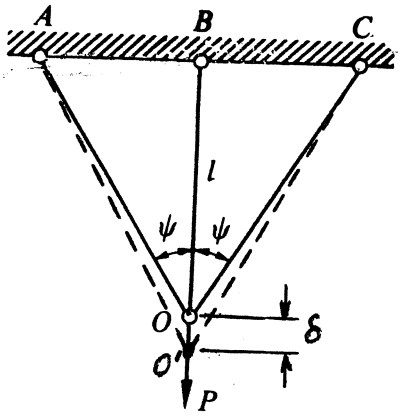
or $$\varepsilon = n + \left(\frac{7n}{3}\right)^n \left(\frac{\sigma_o}{E}\right)^{1-n}$$

When $n = 0.05$ and $\sigma_o/E = 0.002$, the instability strain is

$$\varepsilon = 0.05 + \left(\frac{0.35}{3}\right)^{0.05} (0.002)^{0.95} \cong 0.0525$$

the percentage error being $0.25/(0.0525) \cong 4.8\%$.

1.7.



If plastic instability occurs simultaneously in the three bars, the true strain is n_1 in the central bar and n_2 in the inclined bars at instability. If O is displaced to O' at the point of necking, then from geometry,

$$O'B = \ell e^{n_1},$$

$$O'A = O'C = \ell e^{n_2} \sec\psi .$$

From geometry,

$$\ell^2 \tan^2\psi = AB^2 = O'A^2 - O'B^2$$

$$= \ell^2 (e^{2n_2} \sec^2\psi - e^{2n_1})$$

or $$e^{2n_1} - 1 = (e^{2n_2} - 1)\sec^2\psi$$

or $$\cos\psi = \sqrt{\frac{\exp(2n_2) - 1}{\exp(2n_1) - 1}} .$$

1.8. During a small deformation produced by a vertical load P at O, the strains in the vertical and inclined bars are

$$\varepsilon_1 = \frac{\delta}{\ell}, \quad \varepsilon_2 = \frac{\delta \cos\psi}{\ell \sec\psi} = \frac{\delta}{\ell} \cos^2\psi$$

respectively. When all the bars are plastic, the corresponding stresses are

$$\sigma_1 = Y\left(\frac{E\delta}{Y\ell}\right)^n, \quad \sigma_2 = Y\left(\frac{E\delta}{Y\ell}\cos^2\psi\right)^n$$

Hence the applied load is

$$P = A(\sigma_1 + 2\sigma_2 \cos\psi)$$

or $$\frac{P}{AY} = \left(\frac{E\delta}{Y\ell}\right)^n (1 + 2\cos^{2n+1}\psi)$$

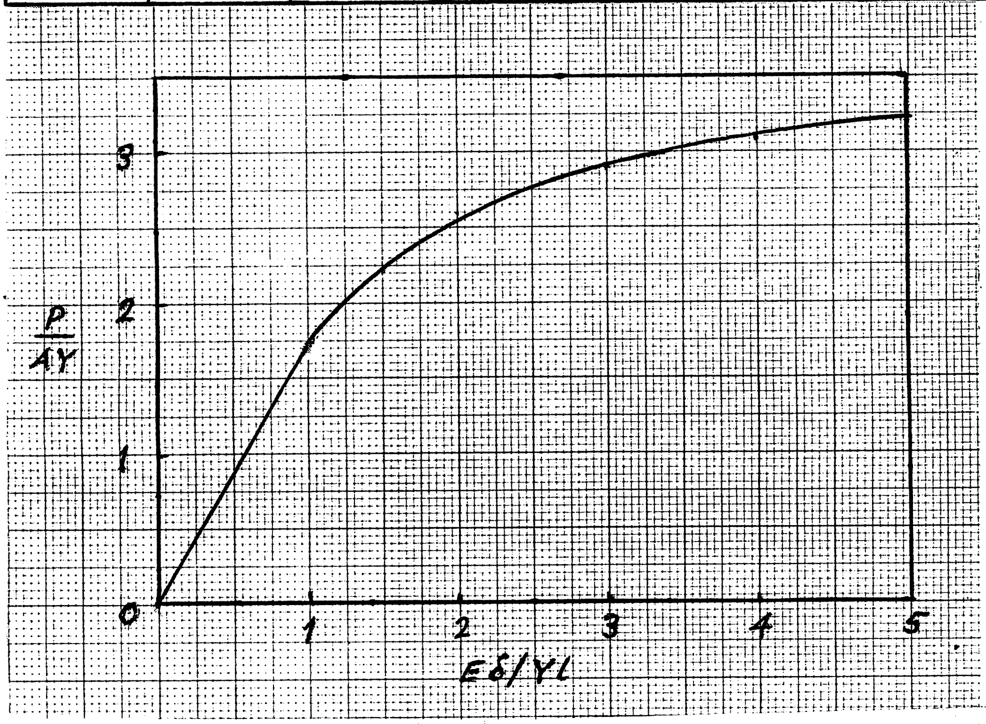
which holds for $\varepsilon_2 \geqslant Y/E$, or $E\delta/Y\ell \geqslant \sec^2\psi$. When only the vertical bar is plastic, the stresses are

$$\sigma_1 = Y\left(\frac{E\delta}{Y\ell}\right)^n, \quad \sigma_2 = \frac{E\delta}{\ell}\cos^2\psi .$$

Hence $$\frac{P}{AY} = \left(\frac{E\delta}{Y\ell}\right)^n + 2\left(\frac{E\delta}{Y\ell}\right)\cos^3\psi$$

which holds for $\varepsilon_1 \geqslant Y/E$ and $\varepsilon_2 \leqslant Y/E$. Therefore, the restriction is $1 \leqslant E\delta/Y\ell \leqslant \sec^2\psi$. The results corresponding to $\psi = \pi/4$ and $n = 0.25$ are

$E\delta/Y\ell$	1.0	1.5	2.0	3.0	4.0	5.0
P/AY	1.707	2.167	2.603	2.881	3.096	3.276

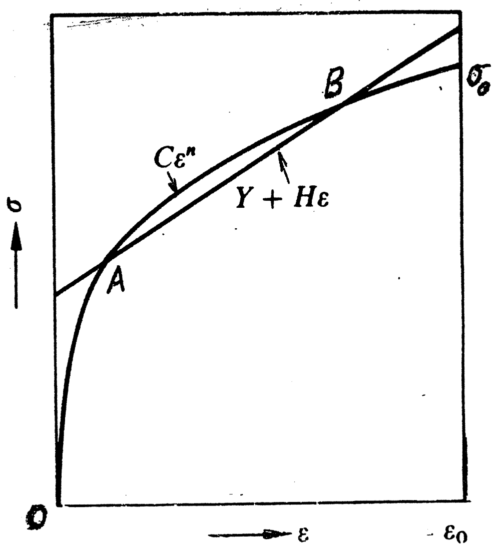


1.9. Equating the total work corresponding to the stress-strain equations,

$$\int_0^{\varepsilon_0} (Y + H\varepsilon)d\varepsilon = C \int_0^{\varepsilon_0} \varepsilon^n \, d\varepsilon$$

or $\quad Y\,\varepsilon_0 + \frac{1}{2} H\,\varepsilon_0 = \frac{C}{1+n}\,\varepsilon_0^{1+n} = \frac{\sigma_0\,\varepsilon_0}{1+n}$

or $\quad 2Y + H\,\varepsilon_0 = 2\,\sigma_0/(1+n)$ (a)



The second condition furnishes the relation

$$(Y + H\varepsilon_0) - C\,\varepsilon_0^{\,n} = 2\left\{ C\left(\frac{\varepsilon_0}{2}\right)^n - \left(Y + \frac{H\varepsilon_0}{2}\right)\right\}$$

or $\quad 3Y + 2H\,\varepsilon_0 = C\,\varepsilon_0^{\,n}(1 + 2^{1-n}) = \sigma_0(1 + 2^{1-n})$ (b)

Solving (a) and (b) for Y and H, we have

$$Y = \sigma_0\left(\frac{3-n}{1+n} - 2^{1-n}\right), \qquad H\,\varepsilon_0 = 2\,\sigma_0\left(2^{1-n} - \frac{2-n}{1+n}\right).$$

The maximum percentage error in the linear approximation over the range AB corresponds to

$$\frac{d}{d\varepsilon}\left(1 - \frac{Y + H\varepsilon}{C\varepsilon^n}\right) = \frac{-H + (Y + H\varepsilon)(n/\varepsilon)}{C\varepsilon^n} = 0$$

or $$\varepsilon = \left(\frac{n}{1-n}\right)\frac{Y}{H} .$$

Hence $$Y + H\varepsilon = \frac{Y}{1-n} = \frac{\sigma_0}{1-n}\left(\frac{3-n}{1+n} - 2^{1-n}\right)$$

and $$C\varepsilon^n = \sigma_0\left(\frac{\varepsilon}{\varepsilon_0}\right)^n = \sigma_0\left(\frac{n}{1-n}\right)^n\left(\frac{Y}{H\ \varepsilon_0}\right)^n$$

$$= \sigma_0\left(\frac{n}{1-n}\right)^n\left(\frac{3-n}{1+n} - 2^{1-n}\right)^n \quad \left\{2\left(2^{1-n} - \frac{2-n}{1+n}\right)\right\}^n .$$

Setting n = 0.3, we get

$$Y + H\varepsilon = \frac{\sigma_0}{0.7}\left(\frac{2.7}{1.3} - 2^{0.7}\right) = \frac{0.4524}{0.7}\ \sigma_0 = 0.6463\ \sigma_0$$

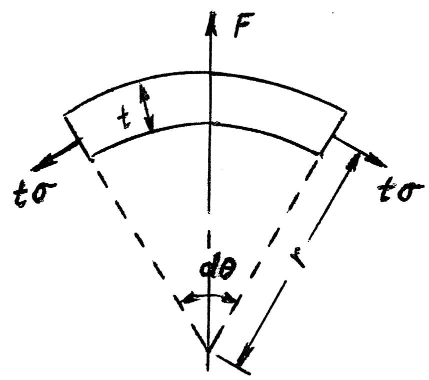
$$C\varepsilon^n = \sigma_0\left(\frac{3}{7}\right)^{0.3}(0.4524)^{0.3}\ \left\{2\left(2^{0.7} - \frac{1.7}{1.3}\right)\right\}^{0.3}$$

$$= \sigma_0\ (0.3060)^{0.3} = 0.7010\ \sigma_0$$

and the percentage error is $\left(1 - \frac{0.6463}{0.7010}\right)(100) = 7.8\%$.

1.10. Let r denote the current mean radius of the ring and t the current thickness. When the angular velocity is ω, the centrifugal force acting in an element of the ring is

$$F = \rho\,\omega^2 r(t\,r\,d\theta) = \rho\,\omega^2 r^2 t\,d\theta .$$



For radial equilibrium, $F = t\,\sigma\,d\theta$, or

$$\sigma = \rho\,\omega^2 r^2 .$$

Hence $$\frac{d\sigma}{\sigma} = 2\left(\frac{d\omega}{\omega} + \frac{dr}{r}\right) .$$

Since $d\omega = 0$ at the onset of instability, we have

$$\frac{d\sigma}{\sigma} = 2\,d\,\varepsilon \quad , \quad \text{or} \quad \frac{d\sigma}{d\varepsilon} = 2\sigma$$

where σ is the uniaxial stress acting in the ring circumferentially, and ε the corresponding strain. Since $d\sigma/d\varepsilon = n\sigma/\varepsilon$ by the given power law, $\varepsilon = n/2$

at instability. The maximum angular velocity is given by

$$\rho\,\omega^2 r_o^2 = \sigma(r_o/r)^2 = C\,\varepsilon^n e^{-2\varepsilon} = C\left(\frac{n}{2}\right)^n e^{-n}$$

1.11. In a thin spherical shell under an internal pressure p, the non-zero principal stresses are each equal to σ. Thus

$$\sigma_\theta = \sigma_\phi = \sigma = \frac{pr}{2t}$$

where r is the current mean radius and t the current thickness. The components of the strain increment are

$$d\varepsilon_r = \frac{dt}{t} = -d\varepsilon, \quad d\varepsilon_\theta = d\varepsilon_\phi = \frac{dr}{r} = \frac{1}{2}\,d\varepsilon \ .$$

Since $dp = 0$ at the onset of instability,

$$\frac{d\sigma}{\sigma} = \frac{dp}{p} + \frac{dr}{r} - \frac{dt}{t} = \frac{3}{2}\,d\varepsilon$$

and the instability condition becomes

$$\frac{d\sigma}{d\varepsilon} = \frac{3}{2}\,\sigma \ .$$

The empirical equation $\sigma = C\varepsilon^n$ gives $n\,\sigma/\varepsilon = 3\,\sigma/2$, or $\varepsilon = \frac{2}{3}\,n$ at instability, and the corresponding thickness and radius are

$$t = t_o e^{-\varepsilon} = t_o \exp\left(-\frac{2}{3}\,n\right)$$

$$r = r_o e^{-\varepsilon/2} = r_o \exp\left(\frac{1}{3}\,n\right).$$

Hence, the bursting pressure is given by

$$\frac{p}{C} = \frac{2\,\sigma\,t}{Cr} = \frac{2\,\sigma\,t_o}{Cr_o}\,e^{-n} = \frac{2\,t_o}{r_o}\left(\frac{2}{3}\,n\right)^n e^{-n} \ .$$

1.12. Let the longitudinal tensile stresses existing in the inner and outer cylinders, having cross-sectional areas A_1 and A_2 respectively, be denoted by σ_1 and σ_2. Then the resultant a ial tension is

$$P = A_1\sigma_1 + A_2\sigma_2 \ .$$

Hence $$\frac{dP}{d\varepsilon} = \left(A_1\,\frac{d\sigma_1}{d\varepsilon} + \sigma_1\,\frac{dA_1}{d\varepsilon}\right) + \left(A_2\,\frac{d\sigma_2}{d} + \sigma_2\,\frac{dA_2}{d}\right)$$

where ε is the longitudinal strain in the composite bar of length ℓ. Since $A_1\ell$ and $A_2\ell$ are constants by the constancy of volume,

$$\frac{dA_1}{A_1} = \frac{dA_2}{A_2} = -\frac{d\ell}{\ell} = -d\varepsilon$$

and A_2/A_1 remains constant during the deformation. At the load maximum, $dP = 0$, giving

$$0 = A_1\left(\frac{d\sigma_1}{d\varepsilon} - \sigma_1\right) + A_2\left(\frac{d\sigma_2}{d\varepsilon} - \sigma_2\right)$$

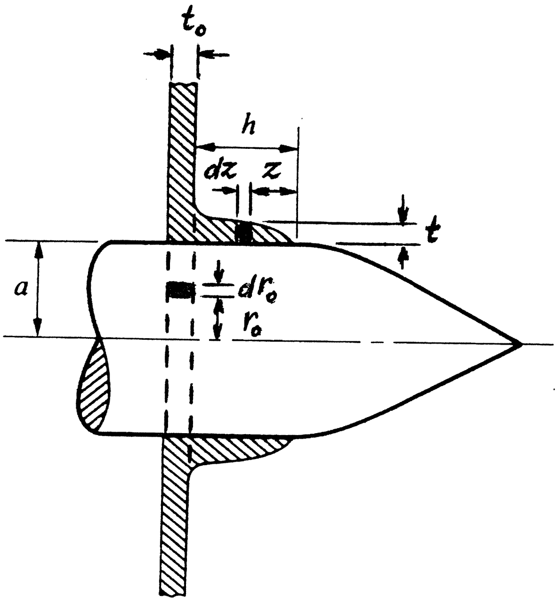
$$= A_1\sigma_1\left(\frac{n_1}{\varepsilon} - 1\right) + A_2\sigma_2\left(\frac{n_2}{\varepsilon} - 1\right)$$

or $$\varepsilon = \frac{A_1\sigma_1 n_1 + A_2\sigma_2 n_2}{A_1\sigma_1 + A_2\sigma_2} = \frac{n_1 + n_2}{2}$$

if $A_1\sigma_1 = A_2\sigma_2$ at instability. Then

$$\frac{A_2}{A_1} = \frac{\sigma_1}{\sigma_2} = \frac{C_1}{C_2}\,\varepsilon^{n_1-n_2} = \frac{C_1}{C_2}\left(\frac{n_1+n_2}{2}\right)^{n_1-n_2} .$$

1.13.



Let r_o be initial radius to an element that is currently at a distance z from the outer edge of the lip. In view of the incompressibility of the material,

$$2\pi r_o t_o dr_o = 2\pi a t dz$$

or $$dz/dr_o = r_o t_o/at .$$

Since the state of stress is uniaxial, the thickness strain is one half in magnitude of the hoop strain. Hence

$$\ell n\left(\frac{t}{t_o}\right) = -\frac{1}{2}\,\ell n\left(\frac{a}{r_o}\right), \text{ or } \frac{t}{t_o} = \sqrt{\frac{r_o}{a}}$$

which gives

$$\frac{dz}{dr_o} = \frac{r_o}{a}\sqrt{\frac{a}{r_o}} = \sqrt{\frac{r_o}{a}}$$

or $$\frac{z}{a} = a^{-3/2}\int_0^{r_o}\sqrt{r_o}\,dr_o = \frac{2}{3}\left(\frac{r_o}{a}\right)^{3/2} .$$

Since $z = h$ when $r_o = a$, we get $h = \frac{2}{3}a$. Also, the thickness variation is given by

$$\frac{t}{t_o} = \left(\frac{r_o}{a}\right)^{1/2} = \left(\frac{3z}{2a}\right)^{1/3} .$$

The plastic work per unit volume of a typical element is

$$E = \int\sigma\,d\varepsilon = C\int_0^{\ell n(a/r_o)}\varepsilon^n d\varepsilon = \frac{C}{1+n}\left(\ell n\frac{a}{r_o}\right)^{1+n} .$$

The total work done during the perforation is

$$W = 2\pi a \int_0^h E\,t\,dz = \frac{2\pi C t_0}{1+n} \int_0^a \left(\ln \frac{a}{r_0}\right)^{1+n} r_0\,dr_0$$

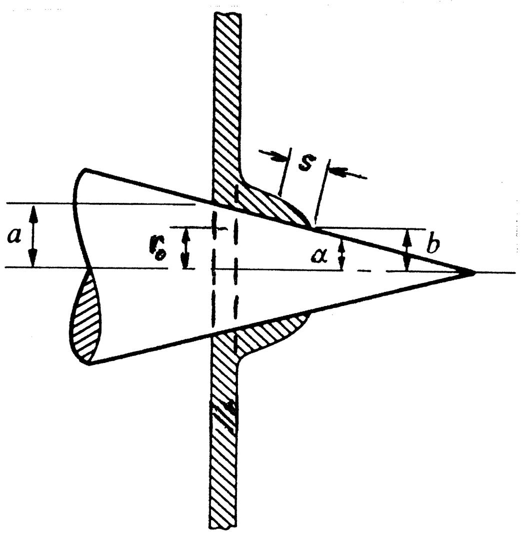
$$= \frac{\pi C t_0 a^2}{(1+n)2^{1+n}} \int_0^\infty e^{-x} x^{1+n}\,dx \quad \left(x = 2 \ln \frac{a}{r_0}\right)$$

or $$W = \frac{\pi C t_0 a^2}{(1+n)2^{1+n}} \Gamma(2+n) = \frac{\pi C t_0 a^2}{2^{1+n}} \Gamma(1+n) .$$

When $n = 0.5$, we have

$$W = \frac{\pi C t_0 a^2}{2^{5/2}} \Gamma\left(\frac{1}{2}\right) = \frac{\pi \sqrt{\pi}\, C t_0 a^2}{4\sqrt{2}} \cong 0.984\, a^2 t_0 C .$$

1.14.



Let s denote the inclined distance of a typical element that was initially at a radial distance r_0 from the axis of symmetry. The incompressibility of the material requires

$$2\pi r_0 t_0\,dr_0 = 2\pi(b + s\sin\alpha)\,t\,ds .$$

Since the state of stress is uniaxial,

$$t_0\sqrt{r_0} = t\sqrt{b + s\sin\alpha} .$$

Hence $\sqrt{b + s\sin\alpha}\,(ds/dr_0) = r_0\sqrt{r_0}$.

The integration of this equation gives

$$(b + s\sin\alpha)^{3/2} - b^{3/2} = r_0^{3/2}\sin\alpha .$$

Since $b + s\sin\alpha = a$ when $r_0 = a$, we have

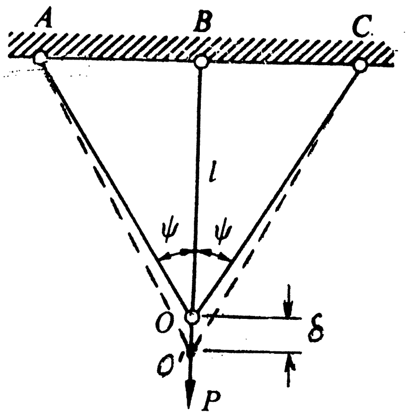
$$a^{3/2} - b^{3/2} = a^{3/2}\sin\alpha, \quad \text{or} \quad b = a(1 - \sin\alpha)^{2/3} .$$

Also, $$\left(1 + \frac{s}{b}\sin\alpha\right)^{3/2} = 1 + \left(\frac{r_0}{b}\right)^{3/2} \sin\alpha .$$

The thickness variation of the perforated plate is therefore given by

$$\frac{t}{t_0} = \sqrt{\frac{r_0}{b}}\left(1 + \frac{s}{b}\sin\alpha\right)^{-1/2} = \sqrt{\frac{r_0}{b}}\left\{1 + \left(\frac{r_0}{b}\right)^{3/2}\sin\alpha\right\}^{-1/3} .$$

1.15.



In the purely elastic range, the stresses are

$$\sigma_1 = E\,\varepsilon_1 = \frac{E\,\delta}{\ell}$$

$$\sigma_2 = E\,\varepsilon_2 = \frac{E\,\delta}{\ell}\cos^2\psi = \sigma_1 \cos^2\psi$$

and the applied load is

$$P = A(\sigma_1 + 2\,\sigma_2 \cos\psi) = A\,\sigma_1\,(1 + 2\cos^3\psi).$$

Hence the elastic stresses become

$$\sigma_1 = \frac{P/A}{1+2\cos^3\psi}\,, \quad \sigma_2 + \frac{(P/A)\cos^2\psi}{1+2\cos^3\psi}\ .$$

Subtracting these stresses from the fully plastic stresses of Prob. 1.8, we obtain the residual stresses. Thus

$$\frac{\sigma_1'}{Y} = \left(\frac{E\,\delta}{Y\ell}\right)^n - \frac{P/AY}{1+2\cos^3\psi} = \left(\frac{E\,\delta}{Y\ell}\right)^n \left\{1 - \frac{1+2\cos^{2n+1}\psi}{1+2\cos^3\psi}\right\}$$

$$\text{or} \qquad \frac{\sigma_1'}{2Y}\sec\psi = -\left(\frac{E\,\delta}{Y\ell}\right)^n \left(\frac{\cos^{2n}\psi - \cos^2\psi}{1+2\cos^3\psi}\right)\ .$$

Since $\sigma_1 + 2\,\sigma_2' \cos\psi = 0$ for the unloaded structure,

$$\sigma_2'/Y = -(\sigma_1/2Y)\sec\psi\ .$$

The residual deflection δ' of point O is obtained by subtracting $(\sigma_1/E)\ell$ from the plastic deflection δ. Thus

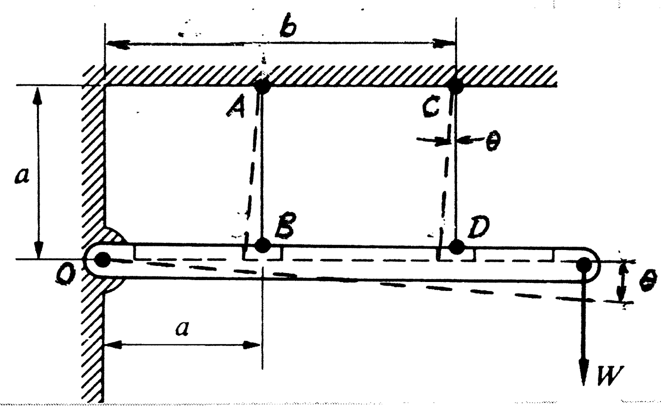
$$\frac{E\,\delta'}{Y\ell} = \frac{E\,\delta}{Y\ell} - \left(\frac{E\,\delta}{Y\ell}\right)^n \left(\frac{1+2\cos^{2n+1}\psi}{1+2\cos^3\psi}\right)\ .$$

When $n = 0.25$, $\psi = \pi/4$ and $E\delta/Y\ell = 3$, we get

$$\frac{\sigma_1'}{Y} = -\sqrt{2}\,(3)^{0.25}\left\{\frac{(0.5)^{0.25} - 0.5}{1+0.707}\right\} = -0.372\ .$$

$$\frac{E\,\delta'}{Y\ell} = 3 - (3)^{0.25}\left\{\frac{1+\sqrt{2}\,(0.5)^{0.25}}{1+0.707}\right\} = 1.312\ .$$

1.16



At the onset of instability, the horizontal rigid bar is inclined at an angle θ. Since the groove is smooth, the stretched wires AB' and CD' must remain perpendicular to the center line of the bar. From geometry, we have

$$AB' = a(\cos\theta + \sin\theta)$$

$$CD' = a\cos\theta + b\sin\theta\ .$$

Since the logarithmic strains in AB and CD are are equal to the strain-hardening indices n and 2n respectively, AB' = a e^n and CD' = a e^{2n}. Hence

$$\frac{a\cos\theta + b\sin\theta}{a(\cos\theta + \sin\theta)} = \frac{CD'}{AB'} = e^n$$

or $$\left(\frac{b}{a} - e^n\right)\sin\theta = (e^n - 1)\cos\theta$$

and $$\frac{b}{a} = e^n + (e^n - 1)\cot\theta \quad .$$

Also, equating AB' to a e^n, we get

$$e^{2n} = (\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta = 1 + \frac{2\tan\theta}{1 + \tan^2\theta}$$

or $$\tan^2\theta - \frac{2\tan\theta}{e^{2n} - 1} + 1 = 0 \quad .$$

Considering the usual situation $e^{2n} < 2$, we obtain the solution

$$\tan\theta = \frac{1 - \sqrt{1 - (e^{2n} - 1)^2}}{e^{2n} - 1} = \frac{1 - e^n\sqrt{2 - e^{2n}}}{e^{2n} - 1}$$

Hence

$$\frac{b}{a} = e^n + \frac{(e^n - 1)(e^{2n} - 1)}{1 - e^n\sqrt{2 - e^{2n}}}$$

Note that the strain-hardening exponent 2n for the wire CD will be less than 0.5 in practical situations. Hence n will be less than 0.25.

1.17. Let (ℓ, m, n) be the direction cosines of the normal to a typical plane containing the given straight line. The orthogonality of the direction (ℓ, m, n) and the given direction requires

$$\ell\sqrt{\frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3}} + n\sqrt{\frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}} = 0$$

or $$\ell^2(\sigma_1 - \sigma_2) = n^2(\sigma_2 - \sigma_3) \quad .$$

The normal stress across this plane is

$$\sigma = \ell^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3 = \ell^2\sigma_1 + n^2\sigma_3 + (1 - \ell^2 - n^2)\sigma_2$$

$$= \ell^2(\sigma_1 - \sigma_2) + n^2(\sigma_3 - \sigma_2) + \sigma_2 = \sigma_2$$

and the shear stress τ across this plane has direction cosines

$$\ell_s = \ell\left(\frac{\sigma_1 - \sigma}{\tau}\right) = \ell\left(\frac{\sigma_1 - \sigma_2}{\tau}\right) , \qquad m_s = m\left(\frac{\sigma_2 - \sigma}{\tau}\right) = 0$$

$$n_s = n\left(\frac{\sigma_3 - \sigma}{\tau}\right) = -\frac{\ell}{\tau}(\sigma_3 - \sigma_2)\sqrt{\frac{\sigma_1 - \sigma_2}{\sigma_2 - \sigma_3}}$$

$$= \frac{\ell}{\tau}\sqrt{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)} \quad .$$

In view of the identify $\ell_s^2 + m_s^2 + n_s^2 = 1$, we have

$$\frac{\ell^2}{\tau^2}(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3) = 1, \quad \text{or} \quad \frac{\ell}{\tau} = \frac{1}{\sqrt{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}} \quad .$$

Hence $\quad \ell_s = \sqrt{\dfrac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3}}, \quad m_s = 0, \quad n_s = \sqrt{\dfrac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}}$

which are also the direction cosines of the given straight line.

1.18. The direction cosines of the normal to the plane are denoted by (ℓ, m, n) with respect to the principal axes. The normal stress across the plane is

$$\sigma = \ell^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3 = \sigma_2$$

$$\text{or} \quad \ell^2\sigma_1 + n^2\sigma_3 = \sigma_2(\ell^2 + n^2), \quad \text{or} \quad \ell^2(\sigma_1 - \sigma_2) = n^2(\sigma_2 - \sigma_3) \; .$$

If the shear stress across the plane is τ, then

$$\tau^2 = (\ell^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2) - \sigma_2^2$$

$$= \ell^2\sigma_1^2 + n^2\sigma_3^2 - (\ell^2 + n^2)\sigma_2^2 = \ell^2(\sigma_1^2 - \sigma_2^2) - n^2(\sigma_2^2 - \sigma_3^2)$$

$$= \ell^2(\sigma_1^2 - \sigma_2^2) - \ell^2(\sigma_1 - \sigma_2)(\sigma_2 + \sigma_3) = \ell^2(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3) \quad .$$

But τ^2 is given as equal to $\frac{1}{4}(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)$. Hence

$$\ell^2 = \frac{1}{4}\left(\frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}\right), \quad n^2 = \frac{1}{4}\left(\frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3}\right), \quad m^2 = 1 - \ell^2 - n^2 = \frac{3}{4} \quad .$$

Only positive values of ℓ, m and n may be taken. Then the direction cosines of the shear stress are

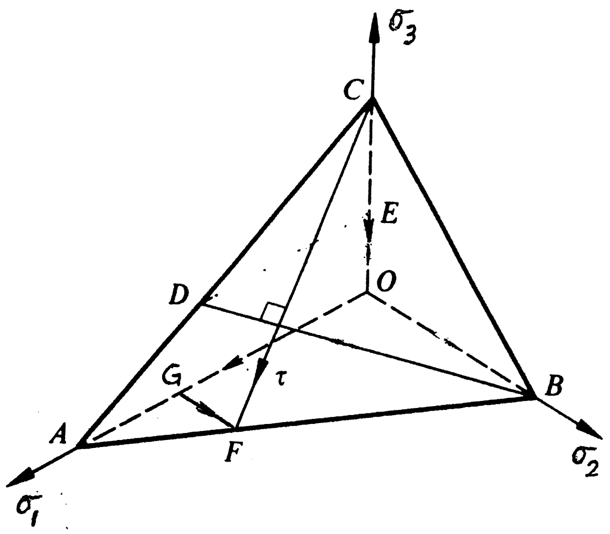
$$\ell_s = \frac{\ell}{\tau}(\sigma_1 - \sigma_2) = \sqrt{\frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3}}, \quad m_s = \frac{m}{\tau}(\sigma_2 - \sigma_2) = 0$$

$$n_s = \frac{n}{\tau}(\sigma_3 - \sigma_2) = -\sqrt{\frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}} \quad .$$

When $\sigma_1 + \sigma_3 = 2\,\sigma_2$, we have $\sigma_1 - \sigma_2 = \sigma_2 - \sigma_3 = \frac{1}{2}(\sigma_1 - \sigma_3)$, and $(\ell_s, m_s, n_s) = (1/\sqrt{2}, 0, -1/\sqrt{2})$, which is precisely the direction of the greatest shear stress.

1.19. The vector CF is equal to the sum of the vectors CO, OG and GF along the σ_3, σ_1 and σ_2 directions respectively. Since CF is in the direction of the shear stress vector,

$$OC : OG : GF = -\,n_s : \ell_s : m_s \quad .$$



It follows from the similar triangles AGF and AOB that

$$\frac{AF}{AB} = \frac{GF}{OB} = \frac{GF}{OC}\frac{OC}{OB} = -\frac{m_s}{n_s}\frac{OC}{OB} \quad .$$

Since $OB : OC = n\ell : \ell m$, we have

$$\frac{AF}{AB} = -\frac{m_s}{n_s}\frac{\ell m}{n\ell} = -\frac{m}{n}\frac{m_s}{n_s} = \frac{m^2}{n^2}\left(\frac{\sigma_2 - \sigma}{\sigma - \sigma_3}\right)$$

in view of the fact that $m_s = m(\sigma_2 - \sigma)/\tau$ and $n_s = n(\sigma_3 - \sigma)/\tau$. Now,

$$\begin{aligned} \sigma_2 - \sigma &= \sigma_2 - (\ell^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3) \\ &= \sigma_2(\ell^2 + n^2) - (\ell^2\sigma_1 + n^2\sigma_3) \\ &= n^2(\sigma_2 - \sigma_3) - \ell^2(\sigma_1 - \sigma_2) \quad . \end{aligned}$$

Since σ_3 is the least principal stress, σ is always greater than σ_3, while $\sigma_2 \gtrless \sigma$ for

$$(\sigma_2 - \sigma_3)/(\sigma_1 - \sigma_2) \gtrless \ell^2/n^2 \quad .$$

This, therefore,is the condition for F dividing AB internally or externally.

1.20. Let (x_o, y_o, z_o) denote the rectangular coordinates of the given point P on the quadric surface. Then

$$x_o = \ell r, \qquad y_o = mr, \qquad z_o = nr$$

where (ℓ, m, n) are the direction cosines of OP. The normal stress across the plane perpendicular to OP is

$$\sigma = \ell^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3 = \frac{\sigma_1 x_o^2 + \sigma_2 y_o^2 + \sigma_3 z_o^2}{r^2} = \pm\frac{c^2}{r^2} \quad .$$

The plus or minus sign in the equation applies according as σ is positive or negative. The equation of the tangent plane at P is

$$\sigma_1 x x_o + \sigma_2 y y_o + \sigma_3 z z_o = \pm c^2$$

or $$\ell\sigma_1 x + m\sigma_2 y + n\sigma_3 z = \pm c^2/r \quad .$$

The length of the perpendicular from the origin to the tangent plane is

$$h = \frac{c^2/r}{\sqrt{\ell^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2}} \quad \frac{c^2}{r|T|}$$

where T is the resultant stress across the plane (ℓ, m, n). The direction cosines of the normal to the tangent plane are proportional to $(\ell\sigma_1, m\sigma_2, n\sigma_3)$. The normal is therefore directed along the resultant stress vector.

1.21. Introducing a factor of proportionality σ, having the dimension of stress, we write

$$[\sigma_{ij}] = \sigma \begin{bmatrix} 2 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & C \end{bmatrix} .$$

Let (ℓ, m, n) be the direction cosines of the normal to the traction-free plane. Across this plane, the components of the resultant stress are

$$\begin{cases} T_x = \ell\, \sigma_x + m\, \tau_{xy} + n\, \tau_{xz} = \sigma(2\ell + 3m + 2n) = 0 \\ T_y = \ell\, \tau_{xy} + m\, \sigma_y + n\, \tau_{yz} = \sigma(3\ell + 2m + n) = 0 \end{cases}$$

$$T_z = \ell\, \tau_{xz} + m\, \tau_{yz} + n\, \sigma_z = \sigma(2\ell + m + Cn) = 0 \quad .$$

The first two equations furnish

$$\frac{\ell}{1} = -\frac{m}{4} = \frac{n}{5} = \lambda \text{ (say)} \quad .$$

Substituting in the relation $\ell^2 + m^2 + n^2 = 1$, we get $42\lambda^2 = 1$, or $\lambda = 1/\sqrt{42}$. Hence

$$\ell = \frac{1}{\sqrt{42}} = 0.154, \quad m = -\frac{4}{\sqrt{42}} = -0.617, \quad n = \frac{5}{\sqrt{42}} = 0.772 \ .$$

The remaining equation ($T_z = 0$) gives

$$C = -\frac{2\ell + m}{n} = -\frac{2-4}{5} = 0.4 \ .$$

1.22. When the principal axes of the stress are taken as the axes of reference, the components of the resultant stress across a plane (ℓ, m, n) are

$$T_x = \ell\, \sigma_1, \quad T_y = m\, \sigma_2, \quad T_z = n\, \sigma_3 \quad .$$

Hence

$$\frac{T_x^2}{\sigma_1^2} + \frac{T_y^2}{\sigma_2^2} + \frac{T_z^2}{\sigma_3^2} = \ell^2 + m^2 + n^2 = 1 \quad .$$

The tip of the stress vector drawn from O will therefore lie on the surface of the ellipsoid

$$\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} + \frac{z^2}{\sigma_3^2} = 1$$

whose principal axes coincide with the coordinate axes.

Let (x_o, y_o, z_o) be the coordinates of the point of intersection of OP with the stress director surface. Then

$$x_o : y_o : z_o = \ell\,\sigma_1 : m\,\sigma_2 : n\,\sigma_3 \quad .$$

The tangent plane to this surface at (x_o, y_o, z_o) has the equation

$$\frac{x\,x_o}{\sigma_1} + \frac{y\,y_o}{\sigma_2} + \frac{z\,z_o}{\sigma_3} = \text{const.}$$

or $\quad \ell\,x + m\,y + n\,z = \text{const.}$

This is a plane parallel to the given plane, specified by the direction of cosines (ℓ, m, n) of its normal.

1.23. The direction cosines of the normal to the given oblique plane are

$$\ell = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \qquad m = n = \sqrt{\frac{1-\ell^2}{2}} = \frac{1}{2} \quad .$$

If θ denotes the angle which the resultant stress vector makes with the normal to the plane, then

$$\cos\theta = \frac{1}{3}\,(2\ell + 2m - n) = \frac{1}{3}\left(\sqrt{2} + \frac{1}{2}\right) = 0.638 \ .$$

The normal and shear stresses across the plane are

$$\sigma = T\,\cos\theta = 135(0.638) = 86.13 \text{ MPa}$$

$$\tau = T\,\sin\theta = 135(0.770) = 103.95 \text{ MPa} \ .$$

Since $\sigma_x = \sigma_y$, $\tau_{xy} = \tau_{yz}$ and $\tau_{zx} = 0$, we have

$$\begin{cases} \ell\,\sigma_x + m\,\tau_{xy} = T_x = \frac{2}{3}\,T = 90 \\ (\ell + n)\tau_{xy} + m\,\sigma = T_y = 90 \end{cases}$$

$$m\,\tau_{yz} + n\,\sigma_z = T_z = -\frac{1}{3}\,T = -45 \quad .$$

Substituting the values of ℓ, m and n into the first two equations, we obtain

$$\sqrt{2}\,\sigma_x + \tau_{xy} = 180 = (\sqrt{2}+1)\tau_{xy} + \sigma_x$$

and the solution, since $\sqrt{2}\,\tau_{xy} = (\sqrt{2}-1)\sigma_x$, is

$$\sigma_x = \frac{180\,\sqrt{2}}{\sqrt{2}+1} = 105.44 \text{ MPa} = \sigma_y$$

$$\tau_{xy} = \frac{180}{(\sqrt{2}+1)^2} = 30.88 \text{ MPa} = \tau_{yz} \quad .$$

The last equation then gives

$$\sigma_z = -(90 + \tau_{yz}) = -120.88 \text{ MPa} \quad .$$

1.24. The direction cosines of the normal to the considered plane with respect to the coordinate axes are

$$\ell = \cos 40^\circ = 0.766\ , \qquad m = \cos 70^\circ = 0.342$$

$$n = \sqrt{1-\ell^2-m^2} = \sqrt{1-0.704} = 0.544\ .$$

The components of the resultant stress across the plane are

$$T_x = \ell\ \sigma_x + m\ \tau_{xy} + n\ \tau_{zx}$$

$$= 0.766\ (64) + 0.342\ (30) + 0.544\ (55) = 89.20\ \text{MPa}$$

$$T_y = \ell\ \tau_{xy} + m\ \sigma_y + n\ \tau_{yz}$$

$$= 0.766\ (30) + 0.342\ (-76) + 0.544\ (-25) = -16.61\ \text{MPa}$$

$$T_z = \ell\ \tau_{zx} + m\ \tau_{yz} + n\ \sigma_z$$

$$= 0.766\ (55) + 0.342\ (-25) + 0.544\ (48) = 59.69\ \text{MPa}\ .$$

The normal and shear stresses across the plane are

$$\sigma = \ell\ T_x + m\ T_y + n\ T_z$$

$$= 0.766\ (89.20) + 0.342\ (-16.61) + 0.544\ (59.69) = 95.12\ \text{MPa}$$

$$\tau = \sqrt{T_x^2 + T_y^2 + T_z^2 - \sigma^2}$$

$$= \sqrt{(89.20)^2 + (16.61)^2 + (59.69)^2 - (95.12)^2} = 52.42\ \text{MPa}\ .$$

Since the components of the normal stress are $\ell\sigma$, $m\sigma$ and $n\sigma$ with respect to the coordinate axes, the direction cosines of the shear stress are

$$\ell_s = \frac{T_x - \ell\sigma}{\tau} = \frac{89.20 - 0.766\ (95.12)}{52.42} = 0.312$$

$$m_s = \frac{T_y - m\sigma}{\tau} = \frac{-16.61 - 0.342\ (95.12)}{52.42} = -0.937$$

$$n_s = \frac{T_z - n\sigma}{\tau} = \frac{59.69 - 0.544\ (95.12)}{52.42} = 0.152\ .$$

1.25. The invariants of the stress tensor σ_{ij} are

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 72.5 - 12.8 = 59.7\ \text{MPa}$$

$$I_2 = -\sigma_x\sigma_y + \tau_{xy}^2 + \tau_{zx}^2 = 72.5\ (12.8) + (62.3)^2 + (45.4)^2 = 6870.5$$

$$I_3 = -\ \sigma_y\ \tau_{zx}^2 = 12.8\ (45.4)^2 = 26382.9\ .$$

The deviatoric stress invariants are

$$J_2 = I_2 + \frac{1}{3} I_1^2 = 6870.5 + \frac{1}{3}(59.7)^2 = 8058.5$$

$$J_3 = I_3 + \frac{1}{3} I_1 I_2 + \frac{2}{27} I_1^3$$

$$= 26382.9 + \frac{1}{3}(59.7)(6870.5) + \frac{2}{27}(59.7)^3 = 178867$$

$$\cos 3\phi = \frac{J_3}{2}\left(\frac{3}{J_2}\right)^{3/2} = \frac{178867}{2}\left(\frac{3}{8058.5}\right)^{1.5} = 0.6424$$

or $$\phi = \frac{50.03}{3} = 16.68° \quad , \quad 2\sqrt{\frac{J_2}{3}} = 2\sqrt{\frac{8058.5}{3}} = 103.65 \quad .$$

Hence, the deviatoric principal stresses are

$$s_1 = 2\sqrt{\frac{J_2}{3}} \cos\phi = 103.65 \cos 16.68° = 99.3 \text{ MPa}$$

$$s_2 = -2\sqrt{\frac{J_2}{3}} \cos\left(\frac{\pi}{3} + \phi\right) = -103.65 \cos 76.68° = -23.9 \text{ MPa}$$

$$s_3 = -2\sqrt{\frac{J_2}{3}} \cos\left(\frac{\pi}{3} - \phi\right) = -103.65 \cos 43.32° = -75.4 \text{ MPa} \quad .$$

The principal stresses are now obtained by adding $I_1/3 = 19.9$ MPa to the corresponding deviatoric components. Thus

$$\sigma_1 = 119.2 \text{ MPa} \quad , \quad \sigma_2 = -4.0 \text{ MPa} \quad , \quad \sigma_3 = -55.5 \text{ MPa} \; .$$

If (ℓ_1, m_1, n_1) are the direction cosines of the algebraically largest principal stress σ_1, then

$$(72.5 - 119.2)\,\ell_1 + 62.3\, m_1 - 45.4\, n_1 = 0$$

$$62.3\,\ell_1 - (12.8 + 119.2)\, m_1 = 0$$

giving the relations

$$m_1 = 0.472\,\ell_1 \quad , \quad n_1 = \frac{-46.7\,\ell_1 + 62.3\, m_1}{45.4} = -0.381\,\ell_1$$

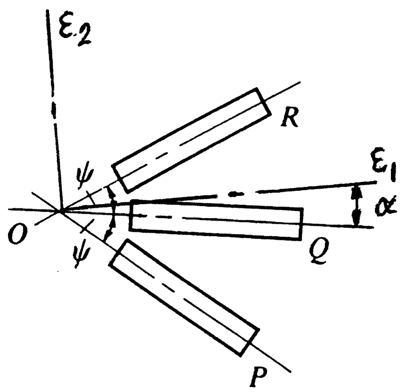
and $$[1 + (0.472)^2 + (0.381)^2]\ell_1^2 = 1 \quad .$$

Hence, $\ell_1 = 0.855$, $m_1 = 0.404$, $n_1 = -0.326$.

1.26. The extensional strains along OP, OQ and OR can be expressed in terms of the principal strains ε_1 and ε_2, and the angles ψ and α. Since the angles made by OP, OQ and OR with the ε_1 direction, measured in the counterclockwise sense, are $-(\psi + \alpha)$, $-\alpha$ and $(\psi - \alpha)$ respectively, we have

$$\varepsilon_P = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2(\psi + \alpha)$$

$$\varepsilon_Q = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\alpha$$



$$\varepsilon_R = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos(\psi - \alpha).$$

These expressions furnish

$$\varepsilon_P - \varepsilon_R = \frac{1}{2}(\varepsilon_1 - \varepsilon_2)[\cos 2(\psi+\alpha) - \cos 2(\psi-\alpha)]$$

$$= -(\varepsilon_1 - \varepsilon_2)\sin 2\psi \sin 2\alpha$$

$$\varepsilon_P + \varepsilon_R - 2\varepsilon_Q = \frac{1}{2}(\varepsilon_1 - \varepsilon_2)[\cos 2(\psi+\alpha) + \cos 2(\psi-\alpha) - 2\cos 2\alpha]$$

$$= -(\varepsilon_1 - \varepsilon_2)(1 - \cos 2\psi)\cos 2\alpha \qquad \text{(a)}$$

and $\tan 2\alpha = \left(\dfrac{1 - \cos 2\psi}{\sin 2\psi}\right)\left(\dfrac{\varepsilon_P - \varepsilon_R}{\varepsilon_P + \varepsilon_R - 2\varepsilon_Q}\right)$

or $\tan 2\alpha = \dfrac{(\varepsilon_P - \varepsilon_R)\tan\psi}{\varepsilon_P + \varepsilon_R - 2\varepsilon_Q}$. (b)

Also, $\varepsilon_P + \varepsilon_R - 2\varepsilon_Q \cos 2\psi = (\varepsilon_1 + \varepsilon_2)(1 - \cos 2\psi)$. (c)

When $\psi = \pi/3$, equations (a), (b) and (c) yield

$$\frac{1}{2}(\varepsilon_1 - \varepsilon_2) = -\frac{1}{3}(\varepsilon_P + \varepsilon_R - 2\varepsilon_Q)\sec 2\alpha$$

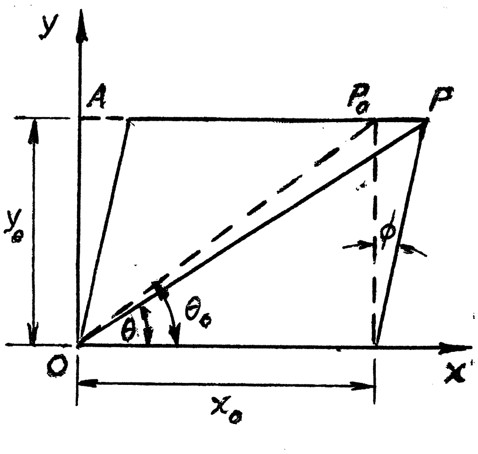
$$= -\frac{1}{3}\sqrt{(\varepsilon_P + \varepsilon_R - 2\varepsilon_Q)^2 + 3(\varepsilon_P - \varepsilon_R)^2}$$

$$\frac{1}{2}(\varepsilon_1 + \varepsilon_2) = \frac{1}{3}(\varepsilon_P + \varepsilon_Q + \varepsilon_R) .$$

In view of the last two relations, the principal strains are

$$\varepsilon_1, \varepsilon_2 = \frac{1}{3}(\varepsilon_P + \varepsilon_R + \varepsilon_R) \pm \frac{1}{3}\sqrt{(\varepsilon_P + \varepsilon_R - 2\varepsilon_Q)^2 + 3(\varepsilon_P - \varepsilon_R)^2} .$$

1.27



A typical particle P_0 moves to P during a simple shear parallel to the x-axis, so that $P_0P = y_0\tan\phi$. From geometry, we have

$$P\,P_0 = AP - AP_0 = y_0(\cot\theta - \cot\theta_0) = y_0\tan\phi$$

or $\quad \cot\theta_0 = \cot\theta - \tan\phi$.

Hence, $-\mathrm{cosec}^2\theta_0 \dfrac{\partial\theta_0}{\partial\theta} = -\mathrm{cosec}^2\theta$,

or $\quad \dfrac{\partial\theta_0}{\partial\theta} = \dfrac{\sin^2\theta_0}{\sin^2\theta}$.

The logarithmic strain associated with OP and OP_0 is

$$\varepsilon = \ln\frac{OP}{OP_0} = \ln\left(\frac{y_0\,\mathrm{cosec}\theta}{y_0\,\mathrm{cosec}\theta_0}\right) = \ln\left(\frac{\sin\theta_0}{\sin\theta}\right) .$$

Hence $\quad \dfrac{\partial\varepsilon}{\partial\theta} = \cot\theta_o \dfrac{\partial\theta_o}{\partial\theta} - \cot\theta = \dfrac{\sin 2\theta_o - \sin 2\theta}{2\sin^2\theta}$.

The directions of maximum extension and contraction correspond to $\partial\varepsilon/\partial\theta = 0$, giving

$$\sin 2\theta = \sin 2\theta_o = \sin 2\left(\pm\frac{\pi}{2} - \theta_o\right), \quad \text{or} \quad \theta = \pm\frac{\pi}{2} - \theta_o \,.$$

The upper sign corresponds to maximum extension, and the lower sign to maximum contraction. Since $\tan\theta = \cot\theta_o$ for these directions,

$$\tan(\theta_o - \theta) = \frac{\tan\theta_o - \tan\theta}{1 + \tan\theta_o \tan\theta} = \frac{1}{2}(\cot\theta - \cot\theta_o) = \frac{1}{2}\tan\phi$$

or $$\theta_o - \theta = \tan^{-1}\left(\frac{1}{2}\tan\phi\right) = \alpha\,, \qquad \theta_o + \theta = \pm\frac{\pi}{2}\,.$$

Hence, $$\theta_o = \pm\frac{\pi}{4} + \frac{\alpha}{2}\,, \qquad \theta = \pm\frac{\pi}{4} - \frac{\alpha}{2}$$

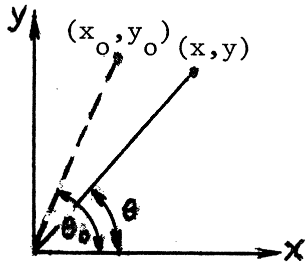
giving the directions of lines that suffer the maximum extension and contraction. The logarithmic strain associated with these directions is

$$\varepsilon = \frac{1}{2}\ln\left(\frac{\sin^2\theta_o}{\sin^2\theta}\right) = \frac{1}{2}\ln\left(\frac{1 - \cos 2\theta_o}{1 - \cos 2\theta}\right) = \frac{1}{2}\ln\left(\frac{1 \pm \sin\alpha}{1 \mp \sin\alpha}\right)$$

$$= \frac{1}{2}\ln\left(\frac{\sec\alpha \pm \tan\alpha}{\sec\alpha \mp \tan\alpha}\right) = \pm\ln(\sec\alpha + \tan\alpha)$$

or $$\varepsilon = \pm\ln(\tan\alpha + \sqrt{1 + \tan^2\alpha}) = \pm\sinh^{-1}(\tan\alpha)\,.$$

1.28. Consider the unit circle $x_o^2 + y_o^2 = 1$ in the initial state. Since $x = c\,x_o$ and $y = d\,y_o$ during the straining, the circle is deformed into the ellipse



$$(x/c)^2 + (y/d)^2 = 1\,.$$

The area of the circle is π, and that of the ellipse in $\pi\,c\,d$. The ratio of the final and initial volumes is therefore equal to $c\,d$, which is unity only for pure shear. The lines which remain unchanged in length correspond to

$$x^2 + y^2 = x_o^2 + y_o^2 = \frac{x^2}{c^2} + \frac{y^2}{d^2}\,.$$

Hence $$\tan^2\beta = \frac{y^2}{x^2} = \frac{d^2}{c^2}\left(\frac{c^2 - 1}{1 - d^2}\right)$$

and $$\tan^2\beta_o = \frac{y_o^2}{x_o^2} = \frac{c^2 y^2}{d^2 x^2} = \frac{c^2}{d^2}\tan^2\beta\,.$$

The straight line initially making an angle θ_o to the x-axis is finally inclined at an angle θ, where

$$\tan\theta = \frac{y}{x} = \frac{d\ y_o}{c\ x_o} = \frac{d}{c}\tan\theta_o \quad .$$

The final inclination θ' of the line that is initially inclined at an angle $\pi/2 + \theta$ to the x-axis is given by

$$\tan\theta' = \frac{d}{c}\tan\left(\frac{\pi}{2} + \theta_o\right) = -\frac{d}{c}\cot\theta_o \quad .$$

Hence $$\tan(\theta' - \theta) = \frac{\tan\theta' - \tan\theta}{1 + \tan\theta'\ \tan\theta} = -\frac{cd}{c^2 - d^2}(\tan\theta_o + \cot\theta_o) \; .$$

The engineering shear strain associated with these two lines is

$$\gamma = \tan\left[\frac{\pi}{2} - (\theta' - \theta)\right] = \cot(\theta' - \theta) = -\frac{c^2 - d^2}{2\,c\,d}\sin 2\,\theta_o \quad .$$

The maximum shear strain is therefore of magnitude $(c^2 - d^2)/2\,c\,d$, corresponding to $\theta = \pm\,\pi/4$.

1.29. From the given expressions for the strain rates,

$$\frac{\partial^2\varepsilon_x}{\partial y^2} = 2\,A\,x^2 \;, \quad \frac{\partial^2\varepsilon_y}{\partial x^2} = 2\,B\,y^2 \;, \quad \frac{\partial^2\gamma_{xy}}{\partial x\ \partial y} = 3\,C\,(x^2 + y^2) \; .$$

The equation of strain rate compatibility then gives

$$2(A\ x^2 + B\ y^2) = 6\,C\,(x^2 + y^2), \quad \text{or } A = B = 3C \; .$$

The velocity field must be determined from the equations

$$\frac{\partial u}{\partial x} = 3\,C\,x^2(x^2 + y^2) \;, \qquad \frac{\partial v}{\partial y} = 3\,C\,y^2(x^2 + y^2)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 2\,C\,xy(x^2 + y^2) \; .$$

The first two equations are integrated to

$$u = C\ x^3\left(\frac{3}{5}\,x^2 + y^2\right) + f(y)$$

$$v = C\ y^3\left(x^2 + \frac{3}{5}\,y^2\right) + g(x)$$

and substitution into the last equation gives

$$f'(y) + g'(x) = 0, \quad \text{or} \quad f'(y) = -g'(x) = D$$

where D is a constant. Then $f(y) = Dy$ and $g(x) = -Dx$, and the velocity field becomes

$$u = C\ x^3\left(\frac{3}{5}\,x^2 + y^2\right) + Dy$$

$$v = C\, y^3 \left[x^2 + \frac{3}{5} y^2 \right] - Dx$$

to within a rigid body translation. If a rigid body rotation of the material as a whole is excluded, then $D = 0$. Assuming this situation, the component of spin is obtained as

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = C\, xy(y^2 - x^2) \quad .$$

1.30. During the straining, the coordinates (x,y) of a typical particle P is changed to $(x+u,\ y+v)$. The displacement of a neighboring particle Q are $u + du$ and $v + dv$, where

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \ , \qquad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \ .$$

The material line element PQ is displaced to P'Q', and the square of the strained line element is $P'Q'^2 = (dx + du)^2 + (dy + dv)^2$. If PQ is parallel to the x-axis $(dy=0)$,

$$P'Q'^2 = \left\{ \left(1 + \frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right\} dx^2 \ .$$

Hence, $$\ln\left(\frac{P'Q'}{PQ} \right) = \frac{1}{2} \ln \left\{ 1 + 2 \frac{\partial u}{\partial x^2} + \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right\}$$

$$\cong \frac{\partial u}{\partial x} + \frac{1}{2} \left\{ \left(\frac{\partial v}{\partial x} \right)^2 - \left(\frac{\partial u}{\partial x} \right)^2 \right\} \qquad \text{(a)}$$

when the third order terms are disregarded. If PQ is parallel to the y-axis, $dx = 0$, and

$$P'Q'^2 = \left\{ \left(1 + \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right\} dy^2 \ .$$

Hence, $$\ln\left(\frac{P'Q'}{PQ} \right) \cong \frac{\partial v}{\partial y} + \frac{1}{2} \left\{ \left(\frac{\partial u}{\partial y} \right)^2 - \left(\frac{\partial v}{\partial y} \right)^2 \right\} . \qquad \text{(b)}$$

If the principal axes of the true strain rate remain fixed in the element during the deformation,

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \cong 0$$

to the first order. Since we are not interested in terms of order higher than the second, we may set $\partial v/\partial x \cong -(\partial u/\partial y)$ in equation (a), and $\partial u/\partial y \cong -(\partial v/\partial x)$ in equation (b), and obtain the principal strains as

$$\varepsilon_1 = \frac{\partial u}{\partial x} + \frac{1}{2} \left\{ \left(\frac{\partial u}{\partial y} \right)^2 - \left(\frac{\partial u}{\partial x} \right)^2 \right\}$$

$$\varepsilon_2 = \frac{\partial v}{\partial y} + \frac{1}{2} \left\{ \left(\frac{\partial v}{\partial x} \right)^2 - \left(\frac{\partial v}{\partial y} \right)^2 \right\} .$$

1.31. The square of a material line element PQ emanating from the particle P in the unstrained state is $ds_o^2 = da_i da_i$, where da_i is the vector representing PQ. The squared length of the line element in the strained state is

$$ds^2 = dx_k dx_k = \frac{\partial x_k}{\partial a_i}\frac{\partial x_k}{\partial a_j} da_i da_j = g_{ij} da_i da_j \quad .$$

Hence, $$\left(\frac{ds}{ds_o}\right)^2 = g_{ij}\left(\frac{da_i}{ds_o}\right)\left(\frac{da_j}{ds_o}\right) = g_{ij} n_i n_j$$

where n_i is the unit vector along PQ in the unstrained state. If PQ is taken along the x-axis, $n_1 = 1$ and $n_2 = n_3 = 0$, and consequently,

$$\left(\frac{ds}{ds_o}\right)^2 = g_{11} , \qquad \text{or} \qquad \frac{ds}{ds_o} = \sqrt{g_{11}} \quad .$$

Similarly, ds/ds_o is equal to $\sqrt{g_{22}}$ when PQ is taken along the y-axis, and equal to $\sqrt{g_{33}}$ when it is taken along the z-axis, in the initial state.

Consider an infinitesimal rectangular parallelepiped in the unstrained state having its edges along the principal axes of g_{ij}. These edges remain orthogonal in the strained state. If the initial and final volumes of the parallelepiped are dV_o and dV respectively, then $dV_o = dV_o\sqrt{g_1 g_2 g_3}$ where g_1, g_2, g_3 are the principal values of g_{ij}. Since $\rho_o dV_o = \rho\, dV$, where ρ_o and ρ are the initial and final densities,

$$\left(\frac{\rho_o}{\rho}\right)^2 = \left(\frac{dV}{dV_o}\right)^2 = g_1 g_2 g_3 = |g_{ij}| = \left|\frac{\partial x_k}{\partial a_i}\frac{\partial x_k}{\partial a_j}\right| \quad .$$

The matrix of the tensor g_{ij} is therefore obtained by premultiplying the matrix of $\partial x_k/\partial a_j$ by its transpose. Since the determinant of a matrix is equal to that of its transpose,

$$\left(\frac{\rho_o}{\rho}\right)^2 = \left|\frac{\partial x_i}{\partial x_j}\right|^2 , \qquad \text{or} \qquad \frac{\rho_o}{\rho} = \left|\frac{\partial x_i}{\partial a_j}\right| \quad .$$

1.32. If the initial and final lengths of a material line element emanating from a typical particle P are denoted by ds_o and ds respectively, then

$$ds_o^2 = da_i\, da_i = \delta_{ij}\, da_i\, da_j$$

$$ds^2 = dx_k\, dx_k = \frac{\partial x_k}{\partial a_i}\frac{\partial x_k}{\partial a_j} da_i\, da_j$$

Hence, $$ds^2 - ds_o^2 = \left(\frac{\partial x_k}{\partial a_i}\frac{\partial x_k}{\partial a_j} - \delta_{ij}\right) da_i\, da_j \quad .$$

Since the right-hand side must be equal to $2\nu_{ij}\, da_i\, da_j$, we have

$$2\gamma_{ij} = \frac{\partial}{\partial a_i}(a_k + u_k)\frac{\partial}{\partial a_j}(a_k + u_k) - \delta_{ij}$$

where u_k is the displacement of P. Since $\partial a_k/\partial a_i = \delta_{ki}$ and $\partial a_k/\partial a_j = \delta_{kj}$, the result is